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CALCULATION OF CRITICAL FORCE DURING BENDING OF A RECTANGULAR ARCH UNDER PRESSURE *

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Abstract. The effectiveness of the variation method proposed for the solution of the considered problem is shown on the problem of determining the stability of a rectangular arch whose ends are closed in different ways. The arch under consideration is under vertical pressure with an intensity that is regularly distributed along its surface. In the presented work, the effect of the geometric and physical parameters of the rectangular arch, which is the research object, on the value of the breaking force was investigated.

Keywords: A variation method, the approximation function, the critical force, the functional, the Rayleigh-Ritz method.

AMS Subject Classification: 70G75, 70K20, 74-06.

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1 Introduction

The possibility of an analytical solution of the equilibrium of arches, that is, the question of elastic stability, was already known to science in the 19th century. The famous scientists G. Kirchhof and R. Klebsch were more thoroughly engaged in these works, they played a major role in the development of the general theory of curved rods. Fundamental scientific theories created by G. Kirchhoff and R. Clebsch were developed by many researchers (Rabotnov, 1977; Alfutov, 1978).

In the present paper (Lalin et al., 2019) a plane round double-hinged arch under the potential dead load is investigated. To describe the stress-strain state and the equilibrium stability the geometrically exact theory is used. According to this theory every point of the bar has two translational degrees of freedom and one rotational, which is independent from the previous two. This paper (Li et al., 2019) investigates the buckling of confined thin-walled functionally graded material (FGM) arch subjected to external pressure. The confined FGM arch buckles in a single-lobe deformation expressed by an admissible radial displacement function. The critical buckling pressure of the confined FGM arch is obtained analytically by establishing the nonlinear equilibrium equations based on the classical thin-walled arch theory. This paper (Wang et al., 2019) describes a study on the compressive arch action of reinforced concrete beams with various levels of horizontal restraints. An analytical model is proposed based on Park's assumption, in which the tension-stiffening effect of steel reinforcement and its slip relative to beam-column

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joints are considered. The model is calibrated by experimental results of beam-column subassemblages and frames subject to column removal scenarios.

Concrete filled steel tubular (CFST) arch has broad application prospects as a high-strength support form in underground excavations, especially in weak and broken surrounding rock.

However, the structural design of CFST arch in underground excavation still relies on experience-based method and lack of quantitative mechanical analysis. This paper (Lu et al., 2020) presents a detailed analysis on the mechanical behavior of CFST arch and proposed an analytical model to construct support characteristic curves (SCC) according to convergence-confinement method. In this paper (Kimura et al., 2020; Gasimov, 2008) a novel method for the shape optimization of tapered arches subjected to in-plane gravity (self-weight) and horizontal loading through compressive internal loading is presented. The arch is discretized into beam elements, and axial deformation is assumed to be small. The curved shape of the tapered arch is discretized into a centroidal B-spline curve with beam elements. Orthodontic wires are integral part of fixed mechanotherapy in orthodontics which are used to facilitate tooth movement. A wide range of materials available that are used to manufacture orthodontic archwires. Selection of suitable wire material increase patient comfort, efficiency of orthodontic mechanics and reduce chair side time. Nowadays esthetic wire materials are available that can be used in esthetically conscious patients. The paper (Kaur et al., 2021) discusses the changing trend in wire materials.

Despite the large number of studies in this area, the problem under discussion cannot be considered definitively solved. In modern technology, the design is widely used, manufactured new materials, the equations of state of which are sufficiently well described relations of the theory of non-linear viscosity. It is important to emphasize that in this case for a number of important tasks, such as stability and bulging, it is necessary to take into account is also geometrical non-linearity. The present work is devoted to such problems.

2 General information about the arch system

The arch system for covering large spans was first proposed by the famous Russian mechanic I.P. Kulibin in 1776, that is, 100 years before construction mechanics was established as a scientific field. He designed and calculated a single-span, arched wooden bridge using the laws of general mechanics. The span of this bridge is 300 meters and it was built over the Neva River in Saint Petersburg. To determine the contour of the axis of the arch, Kulibin experimentally developed the theory of the rope-shaped polygon. The theory was introduced very late in the course of mechanics. Thus, Kulibin was the first to discover the law of interaction of forces in a three-legged statically determined system.

The 30-meter giant bridge model developed by Kulibin was tested and approved by the Russian Academy of Sciences under the influence of 3500 pounds of load.

Euler, an outstanding mathematician of that time, an academician of the Russian Academy of Sciences, checked all the drawings and mathematical calculations of the 30-meter bridge and accepted the obtained results as completely correct.

In order to evaluate the role of Kulibin in bridge construction in the 18th century, it should be noted that the longest wooden bridge with a length of 119 meters was built by the Gruberman brothers in 1778 in Wettingen Abbey based on his calculations.

Arch-shaped devices are widely used as the main structural elements of buildings built for various purposes. According to the static scheme, the types of arches are: three-jointed, two-jointed and jointless (Fig. 1).

Figure 1: Static diagrams of arches

According to the leaning scheme, the arches are as follows (Fig. 2):
1) non-convex arches, in this case, the pressure created under the influence of the force acting in the vertical direction and going in the horizontal direction is transferred to the supports;
2) convex arches, in which case the pressure described above is not transmitted to the supports.

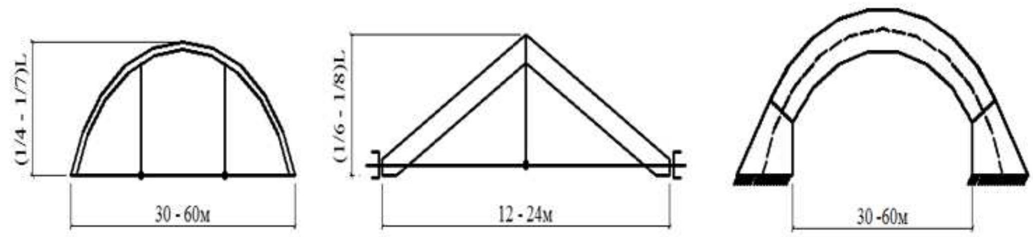

Figure 2: Convex and non-convex arches

Arches are divided according to their shape: straight semi-arched triangular; segment-shaped, when the axes of the arch lie on the common circle; pointed arches, they consist of such semi-arches that the axes of these semi-arches lie on two circles and are connected at a point at a certain angle; polygonal shape (Fig. 3). According to the construction, the types of arches are: full; glued; consisting of farms (Fig. 4).

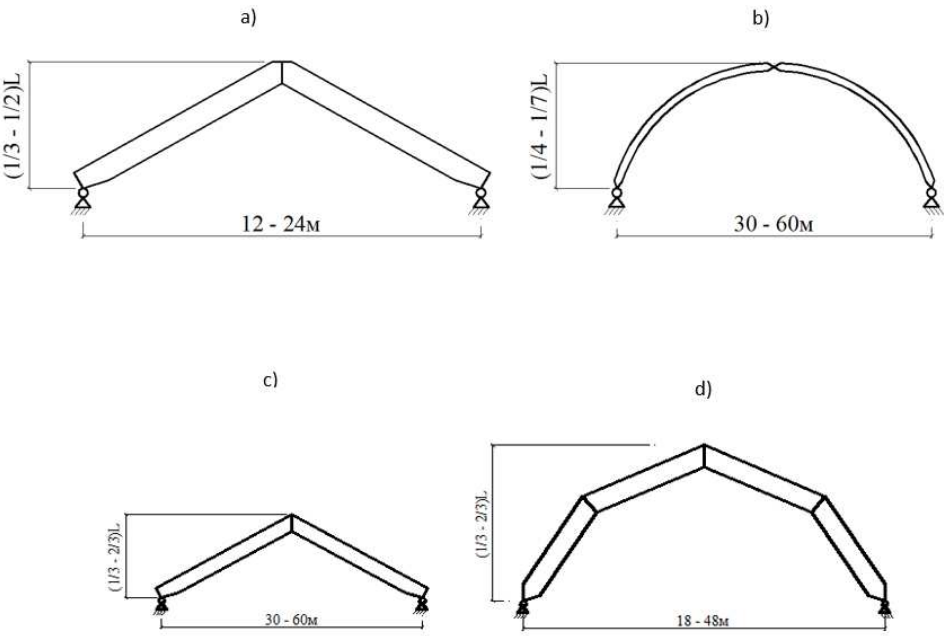

Figure 3: Forms of arches: − triangular; *b*− circular; *c*− pointed; *d*− polygon

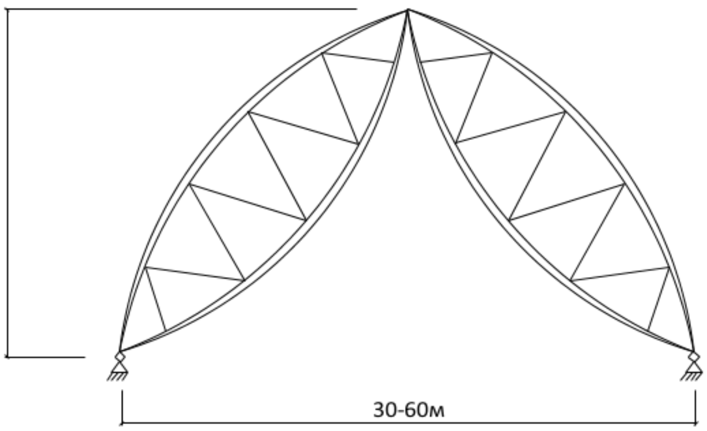

Figure 4: Arch consisting of trusses

Triple arches are the simplest and most common type of arch. The forces generated in them do not depend on deformation and horizontal pressure, the ridge joint is simple. Two-leg arches are used in segmental trusses and glued constructions in special cases. They do not have such an advantage over three-legged arches, but when it is possible to use complete elements, they allow some savings.

Arches that are not subjected to pressure in the horizontal direction are simpler, they consist of only two identical half-arches. Such arches are usually used in fairly tall buildings without vertical walls: sports, mass events and warehouse-type buildings. Arches subjected to pressure in the horizontal direction are used in general types of coverings, such as axial and trussed arches, and rest on walls or columns.

Factory-made arches consisting of glued elements have a wider range of applications, as their shape, dimensions and main characteristics can meet the requirements of the most diverse purpose devices.

Arches made of complete elements used in the construction of buildings can be effectively applied, but their shape, spacing and main characteristics depend on the types of building materials. Arches glued to wood, i.e., factory-made and attached to a flat board, are also widely used ($L = 12 - 16m$).

3 Formulation of the problem

When solving specific problems, greater difficulties of a mathematical nature. This circumstance is connected with the fact that theoretical studies in this area lead to the integration of non-linear regional tasks. Obtaining analytical solutions is very difficult, and impossibility. Therefore, the need arises in the development and application of so important in the applied aspect of the problems of effective approximations, in particular features of variational methods (Amenzadeh et al., 1995). In recent years, the point of view on variational methods has changed significantly in the mechanics of a deformable solid. Possibilities of constructing variational principles of various kinds, i.e. finding the functionals for which the equation given by the Euler equations, proved to be significantly broader, than this was said earlier (Amenzadeh et al., 2010). Reveal the possibility of a sufficiently free choice of independent symbolic function arguments.

The effectiveness of the proposed variation method is shown on the problem of determining the stability of a rectangular arch with non-homogeneous thickness and hinged ends. The presented arch is under vertical pressure with an intensity q, distributed regularly along its surface.

Suppose that the axis of a rectangular arch with hinges at both ends is as follows:

$$\omega = c_0 \eta \sin\left(\frac{\pi z}{l}\right), \tag{1}$$

where c_0 is the axis of elevation of the arch, η is the approximation function, l is the distance between the supports of the arch, and z is the vertical coordinate (Fig. 5).

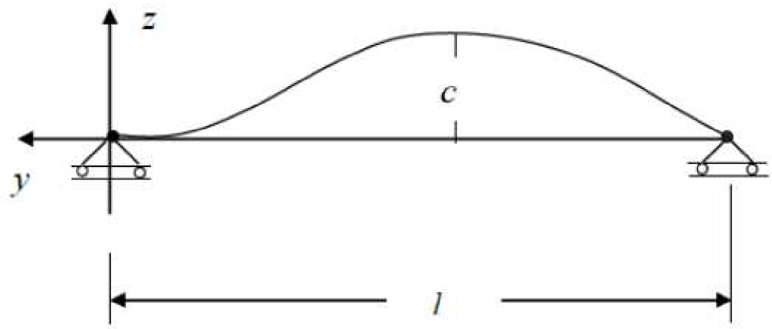

Figure 5: A model of a closed rectangular arch with joints at both ends

It is clear that expression (1) satisfies the boundary conditions of hinged connection of both ends, that is

$$\omega(0) = \omega(l) = 0.$$

The cross-section of the arch is rectangular, its height is $2h$, and its width is b. It is assumed that the arch is geometrically non-linear, that is, it consists of a number of layers n with different thicknesses. Let us denote the thickness of each layer by δ_{k+1}, then

$$\sum_{k=0}^{n-1} \delta_{k+1} = 2h.$$

Let's write the equation of state of the arch in the form of the following equation

$$\varepsilon^v = \frac{\sigma}{E_{k+1}(y)}, a_k \leq y \leq a_{k+1}, \tag{2}$$

where $\sigma$is a stress, E_{k+1}, $[k = 0, 1, ..., (s-1)]$ is the modulus of elasticity of the material of the k-th layer. The modulus of elasticity in each layer depends on the horizontal coordinate y. i.e. $E_{k+1} = E_{k+1}(y)$. In expression (2) the following substitution was made

$$a_k = -h + \sum_{j=0}^{k} \delta_j \quad (\delta_0 = 0).$$

Let's introduce the expression of the functional as follows (Fatullayeva, 2005)

$$J = b \int_{-h}^{h} \int_{0}^{l} \left\{ \dot{\sigma}\dot{\varepsilon} + \frac{1}{2}\sigma\dot{\omega}_{,z} \right\} dydz - \frac{1}{2}\int_{-h}^{h}\int_{0}^{l} \dot{\sigma}\dot{\varepsilon}^v dydz + \int_0^l \dot{q}\dot{\omega}dz. \tag{3}$$

Considering expression (2), the formula of functional (3) takes the following form

$$J = b \int_{-h}^{h} \int_{0}^{l} \left\{ \dot{\sigma}\dot{\varepsilon} + \frac{1}{2}\sigma\dot{\omega}_{,z}^2 \right\} dydz - \frac{1}{2}\int_{0}^{l}\sum_{k=0}^{n-1}\int_{a_k}^{a_{k+1}} \frac{\dot{\sigma}^2}{E_{k+1}(y)} dydz + \int_0^l \dot{q}\dot{\omega}dz\,. \tag{4}$$

The rate of deformation involved in formula (4) is defined as

$$\dot{\varepsilon} = \omega_{,z}\dot{\omega}_{,z} - y\dot{\omega}_{,zz}. \tag{5}$$

Let us define the approximation function and its speed as follows

$$\sigma = E_1\left(\sigma_0^{\nu} + \sigma_1^{v}\left(\frac{2y}{h}\right)\right), \dot{\sigma} = E_1\left(\dot{\sigma}_0^{v} + \dot{\sigma}_1^{v}\left(\frac{2y}{h}\right)\right), \tag{6}$$

where

$$\sigma_0^{\nu} = \sigma_0 \sin\left(\frac{\pi z}{l}\right), \quad \sigma_1^{\nu} = \sigma_1 \sin\left(\frac{\pi z}{l}\right).$$

4 Obtaining the formula for the critical force

The subsequent course of the calculations is as follows: expressions (1), (5), (6) and their corresponding derivatives are written in place of the formula (4) of the functional, and then mathematical calculations are performed. As a result, the following expression for the functional is obtained

$$J = \frac{bhE_1\pi^2}{l}c_0^2\dot{\sigma}_0\eta\dot{\eta} + \frac{2}{3}bh^2E_1c_0\frac{\pi^2}{l}\dot{\sigma}_1\dot{\eta} + \frac{1}{2}bhE_1\frac{\pi^2}{l}c_0^2\dot{\eta}^2\sigma_0 -$$

$$-\frac{bl}{2}E_1^2\dot{\sigma}_0^2\Phi_0 - \frac{4l}{\pi h}bE_1^2\dot{\sigma}_0\dot{\sigma}_1\Phi_1 - \frac{bl}{2h^2}E_1^2\dot{\sigma}_1^2\Phi_2 + \dot{\eta}c_0\frac{2l}{\pi}. \tag{7}$$

This expression of the functional has the following form

$$\Phi_i = \sum_{k=0}^{n-1}\int_{a_k}^{a_{k+1}}\frac{y^i}{E_{k+1}}dy, \quad i = 0, 1, 2.$$

The Rayleigh-Ritz method is used to find the stationary values of functional (7) and the stationary value of the functional is calculated

$$\frac{\partial J}{\partial \dot{\eta}} = 0, \quad \frac{\partial J}{\partial \dot{\sigma}_0} = 0, \quad \frac{\partial J}{\partial \dot{\sigma}_1} = 0.$$

Then such a system of equations is obtained

$$\frac{bhE_1\pi^3}{2l^2}c_0\dot{\sigma}_0\eta + \frac{1}{3}bh^2E_1\frac{\pi^3}{l^2}\dot{\sigma}_1 + \frac{1}{2}bhE_1\frac{\pi^3}{l^2}c_0\dot{\eta}\sigma_0 + 1 = 0, \frac{\pi^2}{l}c_0^2h\eta\dot{\eta} - lE_1\dot{\sigma}_0\Phi_0 - \frac{4l}{\pi h}E_1\dot{\sigma}_1\Phi_1 = 0, \tag{8}$$

$$\frac{2\pi^2}{3l}c_0h^2\dot{\eta} - \frac{4l}{\pi h}E_1\dot{\sigma}_0\Phi_1 - \frac{l}{h^2}E_1\dot{\sigma}_1\Phi_2 = 0.$$

If integrate equations (8) within the initial conditions

$$\eta(0) = 1, \quad \sigma_0(0) = \sigma_1(0) = 0,$$

we get

$$\frac{bhE_1\pi^3}{2l^2}c_0\sigma_0\eta + \frac{1}{3}bh^2E_1\frac{\pi^3}{l^2}\sigma_1 + \frac{1}{2}bhE_1\frac{\pi^3}{l^2}c_0\eta\sigma_0 + q = 0, \frac{\pi^2}{2l}c_0^2h\eta^2 - lE_1\sigma_0\Phi_0 - \frac{4l}{\pi h}E_1\sigma_1\Phi_1 = 0, \tag{9}$$

$$\frac{2\pi^2}{3l}c_0h^2\eta - \frac{4l}{\pi h}E_1\sigma_0\Phi_1 - \frac{l}{h^2}E_1\sigma_1\Phi_2 = 0.$$

From the last two equations of the system of equations (9) we find the parameters σ_0 and σ_1, and substitute them in the first equation. If we introduce the dimensionless quantities

$$\xi = \frac{c_0}{h}\,, \quad \lambda = \frac{h}{l}\,, \quad \tau = \frac{q}{E_1 b}\,, \quad \varphi_0 = \frac{E_1}{h}\Phi_0\,, \quad \varphi_1 = \frac{E_1}{h^2}\Phi_1\,, \varphi_2 = \frac{E_1}{h^3}\Phi_2,$$

then we get the following equation to determine the shear force

$$\tau = \left[-3\pi^7\xi^3\lambda^4\varphi_2\eta^3 + 20\pi^6\xi^2\lambda^4\varphi_1\eta^2 - \frac{4}{3}\pi^7\xi\lambda^4\varphi_0\eta\right]\cdot A^{-1}, \tag{10}$$

where

$$A = 6\pi^2\varphi_0\varphi_2 - 96\varphi_1^2.$$

It is clear that the critical force takes its extremal value when

$$\frac{d\tau}{d\eta} = 0. \tag{11}$$

At this time, a crack appears in the structure of the arch. The value of η_{cr} is found from equation (11) and by substituting this value in expression (10), the critical force is calculated.

Thus, equation (10) is a nonlinear algebraic equation. The quantities involved in it are the geometrical and mechanical-physical parameters of the rectangular arch. Given these parameters, the crunch force is calculated. Depending on the values of the parameters characterizing the material of the arch, it is possible to reduce or increase the breaking force. Then it is possible to obtain the optimal version of the stability of constructions.

5 Numerical results

Let's assume that the thickness of the rectangular arch, which is the object of research, consists of three layers ($n = 3$) and has a periodic structure:

$$E_1 = E_3\,, \quad \delta_1 = \delta_3.$$

In addition, if to introduce the dimensionless quantities

$$\alpha = \frac{E_1}{E_2}\,, \beta = \frac{\delta_2}{\delta_1},$$

then the parameters φ_i ($i = 0\,,1\,,2$) involved in formula (10) will be as follows

$$\varphi_0 = 1\,, \quad \varphi_1 = 0\,, \varphi_2 = \frac{1 + 1,5\beta + 0,75\beta^2 + 0,125\alpha\beta^3}{(1 + 0,5\beta)^3}\,.$$

Using formulas (10), (11) and taking into account the last relations, after simple mathematical transformations, the following expression for the critical force is obtained

$$\tau_{cr} = \frac{8}{81\sqrt{3}}\cdot\frac{\pi^5\lambda^4}{\varphi_2\sqrt{\varphi_2}}\,. \tag{12}$$

If to take $\lambda = 10^{-1}$ in the formula (12), the the numerical values will be found for the critical force dependence on parameters α and β (Figure 6 and 7).

From data of the figures 6 and 7 follows:

- at a given β increase α leads to a decrease in the critical force $\tau_{:@}$. This means that fixing E_1 the increase α is associated with a decrease E_2 the elastic modulus of the second layer, which leads to a decrease in the overall rigidity of the arch, which does not seem rational from the point of view of stability;

- at a given α critical force $\tau_{:@}$ decreases with increasing β. And this means that fixing δ_2 the increase β is associated with a decrease δ_1 the thickness of the extreme layers.

6 Conclusion

The aim of the paper is the development of the theoretical foundations of stability problems in modeling arched structures, which are widely used in the construction of buildings, bridges, railway tunnels, etc. Experiments and calculations have shown that structural elements made of homogeneous materials rarely have properties that meet the requirements of a particular application. However, by combining materials, i.e. by creating and using heterogeneity, it is often possible to achieve a favorable combination of properties, which makes it possible to effectively use structures. The practical value of the results of the work lies in the fact that they can be used in a reliable assessment of the bearing capacity of arch systems. Thus, by combining the properties of the material of the layers and their thickness, it is possible to achieve a more efficient and complete use of the bearing capacity of structural elements. The obtained results of the article can be used in modeling structures, i.e. it is possible to increase (decrease) the value of the critical force.

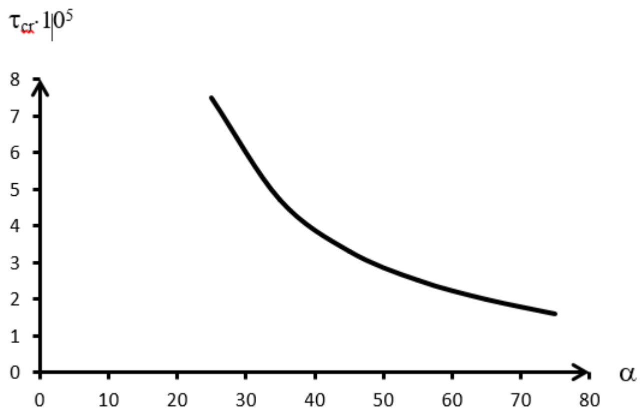

Figure 6: Dependence of the critical force $\tau_{:@}$ from the parameter $\alpha(\beta = 4)$

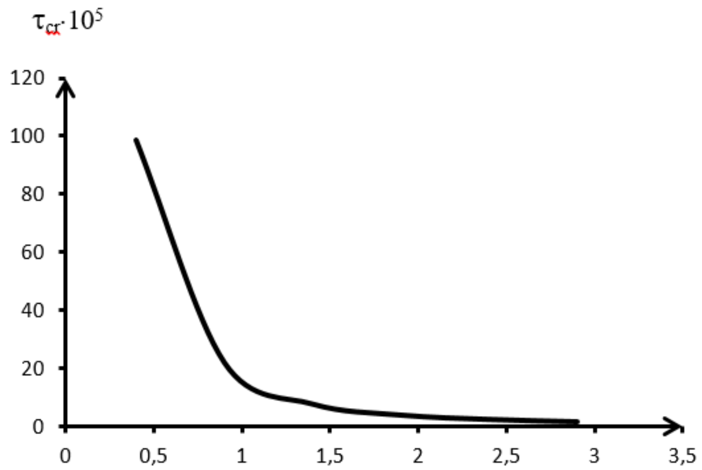

Figure 7: Dependence of the critical force $\tau_{:@}$ from the parameter $\beta(\alpha = 100)$.

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